

NON-COMMUTATIVE q -BINOMIAL FORMULA

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Abstract

In this paper, we found new q -binomial formula for Q -commutative operators. Expansion coefficients in this formula are given by q -binomial coefficients with two bases (q, Q) , determined by Q -commutative q -Pascal triangle. Our formula generalizes all well-known binomial formulas in the form of Newton, Gauss, symmetrical, non-commutative and Binet-Fibonacci binomials. By our non-commutative q -binomial, we introduce q -analogue of function of two non-commutative variables, which could be used in study of non-commutative q -analytic functions and non-commutative q -traveling waves.

1 Introduction

The Newton's Binomial Formula for positive integer n is given in the following form

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k, \quad (1)$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

denotes the binomial coefficients.

The q -analogue of the Newton's Binomial formula is given for the q -analogue of binomial $(x - a)^n$, called the q -binomial.

Definition 1.0.1 *The q -analogue of $(x - a)^n$ is the polynomial*

$$(x - a)_q^n = \begin{cases} 1 & \text{if } n = 0, \\ (x - a)(x - qa)(x - q^2a) \dots (x - q^{n-1}a) & \text{if } n \geq 1 \end{cases}$$

For this q -binomial the Gauss's Binomial formula for **commutative** x and a ($xa = ax$) is written in the following form [1]

$$(x + a)_q^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^{n-k} a^k, \quad (2)$$

where the q -binomial coefficients are

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!},$$

and

$$[n]_q! = [1]_q [2]_q \dots [n]_q, \quad [n]_q = \frac{q^n - 1}{q - 1}.$$

In addition to the Gauss's Binomial formula, the **non-commutative** Binomial formula for the q -**commutative** x and y ($yx = qxy$) is valid [1]

$$(x + y)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}, \quad (3)$$

where q is a number, commuting with x and y ; $xq = qx$ and $yq = qy$.

In the present paper, we derive new type of non-commutative q -binomial for q -binomial with q -commutative entries. We show that this type of formula is naturally related with q -calculus with two bases (Q, q) .

2 Q-commutative q-Binomial Formula

In several physical and mathematical problems we encounter to find a new type of binomial formula. The q -analogue of some known PDE like q -wave and q -Burgers equation include Q -derivative operator D_Q and the dilatation operator M_Q which are non-commutative, but Q -commutative operators. Solutions of these equations are represented as q -binomials in terms of non-commutative operators [2]. Non-commutative versions of q -traveling waves [8] with non-commutative x and t provides non-commutative space-time equations. From another side, extension of quantum groups to two basis (Q, q) require q -calculus with multiple q -numbers. Quantum q -oscillator with such symmetries has been discussed in [7]. It turns out that non-commutative q -binomial formulas are naturally described in terms of such two base calculus.

Firstly, we note that in the standard notation of q -binomial (we using notation from Kac [1])

$$(x + y)_q^n = (x + y)(x + qy)(x + q^2y) \dots (x + q^{n-1}y), \quad n = 1, 2, \dots \quad (4)$$

applied to the noncommutative operators x and y , we should distinguish the direction of multiplication. So we introduce the following notation for two different cases of order [2]

$$(x + y)_{<_q}^n \equiv (x + y)(x + qy)(x + q^2y) \dots (x + q^{n-1}y) \quad (5)$$

and

$$(x + y)_{>q}^n \equiv (x + q^{n-1}y) \dots (x + qy)(x + y). \quad (6)$$

Before introducing the q - Binomial formula for Q -commutative operators, we briefly give the definition of multiple q -numbers :

Multiple q -Number is defined by a basis vector \vec{q} with coordinates q_1, q_2, \dots, q_N as a matrix q -number,

$$[n]_{q_i, q_j} \equiv \frac{q_i^n - q_j^n}{q_i - q_j} = [n]_{q_j, q_i}, \quad (7)$$

which is symmetric.

Theorem 2.0.2 *Let x and y are Q -commutative operators, $yx = Qxy$, then the q -Binomial formula is valid*

$$(x + y)_{<q}^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{Q,q} q^{\frac{k(k-1)}{2}} x^{n-k} y^k, \quad (8)$$

where (Q, q) - binomial coefficients are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{Q,q} = \frac{[n]_{Q,q}!}{[n-k]_{Q,q}! [k]_{Q,q}!}, \quad (9)$$

and the (Q, q) -numbers are

$$[n]_{Q,q} = \frac{Q^n - q^n}{Q - q}. \quad (10)$$

Proof 2.0.3 *Here we give two differen proofs of this theorem. The first one reduces the problem to solution of system of difference equations. The second one is based on the method of mathematical induction.*

To find expansion of q -polynomials in terms of x and y powers we suppose the following expansion

$$(x + y)_{<q}^n = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Q,q} x^{n-k} y^k, \quad (11)$$

where

$$(x + y)_{<q}^n = (x + y)(x + qy)(x + q^2y) \dots (x + q^{n-1}y),$$

and $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Q,q}$ - denote unknown coefficients, depending on k, n, q and Q . Then by definition ,

$$(x + y)_{<q}^{n+1} = (x + y)_{<q}^n (x + q^n y).$$

Expanding both sides

$$\begin{aligned}
\sum_{k=0}^{n+1} \left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\}_{Q,q} x^{n-k+1} y^k &= \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Q,q} x^{n-k} y^k (x + q^n y) \\
&= \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Q,q} x^{n-k} y^k x + \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Q,q} q^n x^{n-k} y^{k+1} \\
&= \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Q,q} Q^k x^{n-k+1} y^k + \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Q,q} q^n x^{n-k} y^{k+1} \\
&= \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Q,q} Q^k x^{n-k+1} y^k + \sum_{k=1}^{n+1} \left\{ \begin{matrix} n \\ k-1 \end{matrix} \right\}_{Q,q} q^n x^{n-k+1} y^k
\end{aligned}$$

from the above equality we have the following recursion formulas :

$$\begin{aligned}
k=0 &\Rightarrow \left\{ \begin{matrix} n+1 \\ 0 \end{matrix} \right\}_{Q,q} = \left\{ \begin{matrix} n \\ 0 \end{matrix} \right\}_{Q,q}, \\
k=n+1 &\Rightarrow \left\{ \begin{matrix} n+1 \\ n+1 \end{matrix} \right\}_{Q,q} = q^n \left\{ \begin{matrix} n \\ n \end{matrix} \right\}_{Q,q}, \\
1 \leq k \leq n &\Rightarrow \left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\}_{Q,q} = Q^k \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Q,q} + q^n \left\{ \begin{matrix} n \\ k-1 \end{matrix} \right\}_{Q,q}, \quad (12)
\end{aligned}$$

where we choose

$$\left\{ \begin{matrix} n \\ b \end{matrix} \right\}_{Q,q} = 0 \text{ if } b < 0 \text{ and } b > n.$$

Suppose the unknown binomial coefficient factor $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Q,q}$ can be written in terms of the known combinatorial coefficient $\left[\begin{matrix} n \\ k \end{matrix} \right]_{Q,q}$ with multiplication factor as

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{Q,q} = q^{t(n,k)} \left[\begin{matrix} n \\ k \end{matrix} \right]_{Q,q}, \quad (13)$$

where $\left[\begin{matrix} n \\ k \end{matrix} \right]_{Q,q}$ is (q, Q) -combinatorial coefficient (9).

Substituting this relation to (12) and using the following relation for q -multiple binomial coefficients, by choosing $q_i = Q$ $q_j = q$,

$$\begin{aligned}
\left[\begin{matrix} n \\ k \end{matrix} \right]_{q_i, q_j} &= \frac{q_j^k [n-1]_{q_i, q_j}!}{[k]_{q_i, q_j}! [n-k-1]_{q_i, q_j}!} + \frac{q_i^{n-k} [n-1]_{q_i, q_j}!}{[n-k]_{q_i, q_j}! [k-1]_{q_i, q_j}!} \\
&= q_j^k \left[\begin{matrix} n-1 \\ k \end{matrix} \right]_{q_i, q_j} + q_i^{n-k} \left[\begin{matrix} n-1 \\ k-1 \end{matrix} \right]_{q_i, q_j}, \quad (14)
\end{aligned}$$

$$= q_i^k \left[\begin{matrix} n-1 \\ k \end{matrix} \right]_{q_i, q_j} + q_j^{n-k} \left[\begin{matrix} n-1 \\ k-1 \end{matrix} \right]_{q_i, q_j}, \quad (15)$$

we have following expression

$$Q^k q^{t(n+1,k)} \begin{bmatrix} n \\ k \end{bmatrix}_{Q,q} + q^{n+1-k+t(n+1,k)} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{Q,q} = Q^k q^{t(n,k)} \begin{bmatrix} n \\ k \end{bmatrix}_{Q,q} + q^{n+t(n,k-1)} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{Q,q}$$

By equating terms with the same power of q and Q , we obtain two difference equations

$$\begin{aligned} t(n+1, k) &= t(n, k) \\ t(n, k) &= t(n, k-1) + k - 1 \end{aligned} \quad (16)$$

with the initial conditions

$$t(0, 0) = t(1, 0) = t(1, 1) = 0. \quad (17)$$

From the first equation for $k = 0$ we have $t(n+1, 0) = t(n, 0)$. So, if $n = 1 \Rightarrow t(2, 0) = t(1, 0) = 0$, which means that $t(n, 0) = 0$. By using the second equation we easily write

$$\begin{aligned} t(n, 1) &= t(n, 0) = 0 \\ t(n, 2) &= t(n, 1) + 1 = 1 \end{aligned}$$

$$\begin{aligned} t(n, 3) &= t(n, 2) + 2 = 1 + 2 \\ t(n, 4) &= t(n, 3) + 2 = 1 + 2 + 3 \end{aligned}$$

...

$$t(n, k) = 1 + 2 + 3 + \dots + (k-1) = \frac{k(k-1)}{2}.$$

Therefore, the solution of the above system is

$$t(n, k) = \frac{k(k-1)}{2}. \quad (18)$$

Hence, we obtain the q - Binomial formula for Q -commutative x and y in the form

$$(x + y)_{<q}^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{Q,q} q^{\frac{k(k-1)}{2}} x^{n-k} y^k, \quad (19)$$

where $yx = Qxy$, and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{Q,q} = \frac{[n]_{Q,q}!}{[n-k]_{Q,q}! [k]_{Q,q}!}, \quad [n]_{Q,q} = \frac{Q^n - q^n}{Q - q}.$$

■

By substituting this relation into equation (20) we have desired result

$$\begin{aligned}
(x+y)_q^{n+1} &= \begin{bmatrix} n \\ 0 \end{bmatrix}_{Q,q} x^{n+1} + \begin{bmatrix} n \\ n \end{bmatrix}_{Q,q} q^{\frac{n(n+1)}{2}} y^{n+1} + \sum_{k=1}^n \begin{bmatrix} n \\ k \end{bmatrix}_{Q,q} q^{\frac{k(k-1)}{2}} x^{n-k+1} y^k \\
&= \sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{Q,q} q^{\frac{k(k-1)}{2}} x^{n-k+1} y^k.
\end{aligned} \tag{21}$$

Example: We consider the Q - derivative operator $D_Q = \frac{M_Q - 1}{x(Q-1)}$, where $M_Q = Q^x \frac{d}{dx}$ is the Q -dilatation operator. These operators are Q -commutative

$$D_Q M_Q = Q M_Q D_Q.$$

Then according to the theorem, we have the binomial expansion

$$\begin{aligned}
(M_Q + D_Q)_{<_q}^n &= (M_Q + D_Q)(M_Q + q D_Q)(M_Q + q^2 D_Q) \dots (M_Q + q^{n-1} D_Q) \\
&= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{Q,q} q^{\frac{k(k-1)}{2}} M_Q^{n-k} D_Q^k.
\end{aligned} \tag{22}$$

Example: The unitary operator $D(\alpha) = e^{\alpha a^\dagger - \bar{\alpha} a}$ for the Heisenberg-Weyl group is the generating operator for Coherent states

$$|\alpha\rangle = D(\alpha)|0\rangle,$$

where α is complex parameter. These operators satisfy the relation

$$D(\alpha) D(\beta) = e^{2i\Im(\alpha\bar{\beta})} D(\beta) D(\alpha),$$

hence they are Q -commutative with $Q = e^{2i\Im(\alpha\bar{\beta})}$. Therefore, applying the above theorem we have the next operator q -Binomial expansion

$$(D(\beta) + D(\alpha))_{<_q}^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{Q,q} q^{\frac{k(k-1)}{2}} D^{n-k}(\beta) D^k(\alpha). \tag{23}$$

If we apply this expansion to vacuum state $|0\rangle$, then we get expansion in superposition of Coherent states

$$\begin{aligned}
(D(\beta) + D(\alpha))_{<_q}^n |0\rangle &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{Q,q} q^{\frac{k(k-1)}{2}} D^{n-k}(\beta) |k\alpha\rangle \\
&= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{Q,q} q^{\frac{k(k-1)}{2}} e^{i\delta} |(n-k)\beta + k\alpha\rangle,
\end{aligned} \tag{24}$$

where

$$\delta = 2\Im[(n-k)k\alpha\bar{\beta}].$$

In the above theorem we have derived Q -commutative (5) q -binomial formula for ordered product $<_q$. Now we like to construct similar formula for opposite ordered product $>_q$ (6).

Proposition 2.0.5 *The following relation for two different ordered products is valid*

$$\prod_{k=0}^N (x + q^k y)_{<_q} = \prod_{k=0}^N (x + Q^{N-2k} q^k y)_{>_q}, \quad (25)$$

where $yx = Qxy$ and

$$\begin{aligned} \prod_{k=0}^N (x + q^k y)_{<_q} &= (x + y)(x + qy)(x + q^2 y) \dots (x + q^N y) = (x + y)_{<_q}^{N+1}, \\ \prod_{k=0}^N (x + Q^{N-2k} q^k y)_{>_q} &= (x + q^N y)(x + q^{N-1} y) \dots (x + qy)(x + y). \end{aligned}$$

Proof 2.0.6 *This formula can be proved by the method of mathematical induction.*

$$\begin{aligned} N = 1 \Rightarrow \prod_{k=0}^1 (x + q^k y)_{<_q} &= (x + y)(x + qy) = x^2 + qQ^{-1}yx + Qxy + qy^2 \\ &= (x + Q^{-1}qy)(x + Qy) = \prod_{k=0}^1 (x + Q^{1-2k} q^k y)_{>_q} \end{aligned}$$

and we suppose that the formula is true for some N . Let us show that it is also valid for $N + 1$:

$$\begin{aligned} \prod_{k=0}^{N+1} (x + q^k y)_{<_q} &= \prod_{k=0}^N (x + q^k y)_{<_q} (x + q^{N+1} y) \\ &= \prod_{k=0}^N (x + Q^{N-2k} q^k y)_{>_q} (x + q^{N+1} y) \\ &= (x + Q^{-N} q^N y) \dots (x + Q^{(N-2)} qy) (x + Q^N y) (x + q^{N+1} y). \end{aligned}$$

By using equality

$$(x + q^m y)(x + q^k y) = (x + Q^{-1} q^k y)(x + Qq^m y),$$

we move the last term to the left-end by commutating with every term of the product

$$\begin{aligned} \prod_{k=0}^{N+1} (x + q^k y)_{<_q} &= (x + Q^{-N} q^N y) \dots (x + Q^{(N-2)} qy) (x + Q^{-1} q^{N+1} y) (x + Q^{N+1} y) \\ &= (x + Q^{-N} q^N y) \dots (x + Q^{-2} q^{N+1} y) (x + Q^{(N-1)} qy) (x + Q^{N+1} y) \\ &= \dots \\ &= (x + Q^{-(N+1)} q^{N+1} y) (x + Q^{-(N-1)} q^N y) \dots (x + Q^{N+1} y) \\ &= \prod_{k=0}^{N+1} (x + Q^{N+1-2k} q^k y)_{>_q}. \end{aligned}$$

Proposition 2.0.7 For $q = 1$ we have the following relation

$$(x + y)^n = (x + y)_{<\tilde{Q}}^n, \quad (26)$$

where

$$(x + y)_{<\tilde{Q}}^n = (x + Q^{-(n-1)}y)(x + Q^{-(n-3)}y) \dots (x + Q^{(n-3)}y)(x + Q^{(n-1)}y)$$

is non-commutative binomial in symmetrical calculus case.

Proof 2.0.8 By mathematical induction, for $n = 1$, it is obvious. Suppose we have

$$(x + y)^n = (x + y)_{<\tilde{Q}}^n,$$

for arbitrary n . Then for $n + 1$ we have

$$\begin{aligned} (x + y)^{n+1} &= (x + y)^n (x + y) = (x + y)_{<\tilde{Q}}^n (x + y) \\ &= (x + Q^{-(n-1)}y)(x + Q^{-(n-3)}y) \dots (x + Q^{(n-3)}y)(x + Q^{(n-1)}y)(x + y) \\ &= (x + Q^{-(n-1)}y)(x + Q^{-(n-3)}y) \dots (x + Q^{(n-3)}y)(x + Q^{(n-1)}y)(x + Q^{(n)}y) \\ &= (x + Q^{-(n-1)}y)(x + Q^{-(n-3)}y) \dots (x + Q^{(n-2)}y)(x + Q^{(n-2)}y)(x + Q^{(n)}y) \\ &= \dots \\ &= (x + Q^{-n}y)(x + Q^{-(n-2)}y) \dots (x + Q^{(n-2)}y)(x + Q^n y) = (x + y)_{<\tilde{Q}}^{n+1}. \end{aligned}$$

We summarize our results in the next q -binomial formula for Q -commutative operators x and y :

$$\begin{aligned} (x + y)_{<q}^N &= \prod_{k=0}^{N-1} (x + q^k y)_{<q} = (x + y)(x + qy)(x + q^2 y) \dots (x + q^{N-1} y) \\ &= \sum_{k=0}^N \begin{bmatrix} N \\ k \end{bmatrix}_{q,Q} q^{\frac{k(k-1)}{2}} x^{N-k} y^k, \end{aligned} \quad (27)$$

where $yx = Qxy$.

Proposition 2.0.9 Two opposite ordered q -binomials are related by formula

$$(x + y)_{<q}^N = (x + Q^{N-1}y)_{>\frac{q}{Q^2}}^N, \quad (28)$$

where $yx = Qxy$.

Proof 2.0.10 From equation (25) we have

$$\prod_{k=0}^N (x + q^k y)_{<q} = \prod_{k=0}^N (x + Q^{N-2k} q^k y)_{>q} = \prod_{k=0}^N (x + (\frac{q}{Q^2})^k Q^N y)_{>q} = (x + Q^N y)_{>\frac{q}{Q^2}}^{N+1};$$

$$(x + y)_{<q}^{N+1} = (x + Q^N y)_{>\frac{q}{Q^2}}^{N+1} \Rightarrow (x + y)_{<q}^N = (x + Q^{N-1} y)_{>\frac{q}{Q^2}}^N.$$

Proposition 2.0.11 For $yx = Qxy$ we obtain the following relation

$$(x + y)_{>q}^N = \prod_{k=0}^{N-1} (x + q^k y)_{>q} = \sum_{k=0}^N \begin{bmatrix} N \\ k \end{bmatrix}_{qQ^2, Q} (qQ^2)^{\frac{k(k-1)}{2}} x^{N-k} \left(\frac{y}{Q^{N-1}} \right)^k. \quad (29)$$

Proof 2.0.12 We start from relation between direction of two multiplication rules (25)

$$\begin{aligned} \prod_{k=0}^{N-1} (x + q^k y)_{<q} &= \prod_{k=0}^{N-1} (x + Q^{N-1-2k} q^k y)_{>q} \\ &= \prod_{k=0}^{N-1} (x + \left(\frac{q}{Q^2}\right)^k Q^{N-1} y)_{>q}. \end{aligned} \quad (30)$$

By choosing $Q^{N-1}y \equiv z \Rightarrow y = \frac{z}{Q^{N-1}}$, the above equation becomes

$$\prod_{k=0}^{N-1} \left(x + \left(\frac{q}{Q^2} \right)^k z \right)_{>q} = \prod_{k=0}^{N-1} \left(x + q^k \frac{z}{Q^{N-1}} \right)_{<q}.$$

Let us call $\frac{q}{Q^2} \equiv q_1$, then according to Proposition 2.0.10

$$\begin{aligned} \prod_{k=0}^{N-1} (x + q_1^k z)_{>q_1} &= (x + z)_{>q_1}^N = \prod_{k=0}^{N-1} \left(x + (q_1 Q^2)^k \frac{z}{Q^{N-1}} \right) = \left(x + \frac{z}{Q^{N-1}} \right)_{<q_1 Q^2}^N \\ &= \sum_{k=0}^N \begin{bmatrix} N \\ k \end{bmatrix}_{q_1 Q^2, Q} (q_1 Q^2)^{\frac{k(k-1)}{2}} x^{N-k} \left(\frac{z}{Q^{N-1}} \right)^k. \end{aligned} \quad (31)$$

Relation $yx = Qxy$ implies $zx = Qxz$, this is why, if we replace $q_1 \rightarrow q$ and $z \rightarrow y$, we obtain the required result. \blacksquare

Finally, q -Binomial formulas for Q -commutative operators x and y with different order are summarized in equations (27) and (29).

Now we are going to show that all known binomial formulas like Gauss Binomial formula etc. are particular cases of our non-commutative binomial formula.

2.1 Special Cases

Let us consider some particular cases of this generalized Q commutative q -Binomial formula:

(i) for $Q = 1$, which means commutative x and y , this formula becomes the Gauss Binomial formula

$$(x + y)_q^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^{n-k} y^k,$$

where $yx = xy$.

(ii) for Q -commutative x and y ($yx = Qxy$) and $q = 1$ we have Non-commutative Binomial formula

$$(x + y)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_Q x^{n-k} y^k.$$

(iii) for $Q = \frac{1}{q}$, we obtain the symmetrical binomial formula

$$(x + y)_q^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{\bar{q}} q^{\frac{k(k-1)}{2}} x^{n-k} y^k,$$

where $yx = \frac{1}{q}xy$, and the symmetrical q -number is defined as

$$[n]_{\bar{q}} = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

(iv) for $q = Q \Rightarrow \lim_{Q \rightarrow q} [n]_{q,q} = nq^{n-1}$, and the formula transforms to the following one

$$(x + y)_q^n = \sum_{k=0}^n \binom{n}{k} q^{k(n-\frac{k+1}{2})} x^{n-k} y^k,$$

where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ standard Newton binomials.

(v) By choosing $q = -\frac{1}{\varphi}$ and $Q = \varphi$, where φ is the Golden ratio, we obtain the Binet-Fibonacci Binomial formula for Golden Ratio non-commutative plane ($yx = \varphi xy$) [7]

$$\begin{aligned} (x + y)_{-\frac{1}{\varphi}}^n &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{\varphi, -\frac{1}{\varphi}} \left(-\frac{1}{\varphi}\right)^{\frac{k(k-1)}{2}} x^{n-k} y^k \\ &= \sum_{k=0}^n \frac{F_n!}{F_k! F_{n-k}!} \left(-\frac{1}{\varphi}\right)^{\frac{k(k-1)}{2}} x^{n-k} y^k, \end{aligned} \quad (32)$$

where F_n are Fibonacci numbers, and q -binomial coefficients become Fibonomial.

(vi) The above formula can be compared with the following general commutative (Q, q) -binomial formula. Let $xy = yx$ and

$$(x + y)_{q,Q}^n = \begin{cases} 1 & \text{if } n = 0, \\ (x + q^{n-1}y)(x + q^{n-2}Qy) \dots (x + qQ^{n-2}y)(x + Q^{n-1}y) & \text{if } n \geq 1 \end{cases}$$

then the binomial formula is valid

$$(x + y)_{q,Q}^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{q,Q} (qQ)^{\frac{k(k-1)}{2}} x^{n-k} y^k.$$

We notice that in this formula x and y are commutative variables, while (q, Q) -binomial coefficients are the same with non-commutative binomial formula.

3 Conclusions

In conclusion we mention one more possible application of our non-commutative q -binomial expansion. We introduce q -function of two variables. If

$$f(x) = \sum_{n=0}^{\infty} c_n x^n,$$

then q -function of two variables x and y we define as

$$f(x+y)_q = \sum_{n=0}^{\infty} c_n (x+y)_q^n. \quad (33)$$

These functions appear in our recent studies on q -analytic functions [9] and q -traveling waves [8]. If we take into account non-commutative binomial formulas derived in this paper we can extend our results [8] and [9] to the q -function of non-commutative (Q -commutative) variables x and y . According to this we can define non-commutative q -analytic functions and non-commutative q -traveling waves. These questions are under the study.

Here we just briefly discuss the case of non-commutative q -exponential function.

Definition 3.0.1 (q, Q) analogues of exponential function are defined as

$$\begin{aligned} e_{q,Q}(x) &\equiv \sum_{n=0}^{\infty} \frac{1}{[n]_{q,Q}!} x^n, \\ E_{q,Q}(x) &\equiv \sum_{n=0}^{\infty} \frac{1}{[n]_{q,Q}!} q^{\frac{n(n-1)}{2}} x^n. \end{aligned} \quad (34)$$

Proposition 3.0.2 For Q -commutative operators x and y , ($yx = Qxy$), we have the following factorization of q -exponential function $e_{q,Q}$,

$$e_{q,Q}(x+y)_{<q} = e_{q,Q}(x)E_{q,Q}(y),$$

Proof 3.0.3

$$\begin{aligned} e_{q,Q}(x+y)_{<q} &= \sum_{N=0}^{\infty} \frac{(x+y)_{<q}^N}{[N]_{q,Q}!} \\ &= \sum_{N=0}^{\infty} \frac{1}{[N]_{q,Q}!} \sum_{k=0}^N \begin{bmatrix} N \\ k \end{bmatrix}_{q,Q} q^{\frac{k(k-1)}{2}} x^{N-k} y^k \\ &= \sum_{N=0}^{\infty} \sum_{k=0}^N \frac{1}{[N-k]_{q,Q}! [k]_{q,Q}!} q^{\frac{k(k-1)}{2}} x^{N-k} y^k. \end{aligned}$$

By choosing $N - k \equiv s$,

$$\begin{aligned} e_{q,Q}(x+y)_{<q} &= \left(\sum_{s=0}^{\infty} \frac{1}{[s]_{q,Q}!} x^s \right) \left(\sum_{k=0}^{\infty} \frac{1}{[k]_{q,Q}!} q^{\frac{k(k-1)}{2}} y^k \right) \\ &= e_{q,Q}(x) E_{q,Q}(y). \end{aligned} \tag{35}$$

■

For $q = 1$, the above exponential functions reduce to the Jackson exponential functions and our proposition gives factorization of this function for Q -commutative argument.

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References

- [1] V. Kac and P. Cheung, Quantum Calculus, Springer, New York, 2002.
- [2] S.Nalci and O.K.Pashaev, q-Analog of shock soliton solution, J.Phys.A:Math.Theor.43, 445205, 2010.
- [3] H. Exton, q-Hypergeometric Functions and Applications, John Wiley and Sons, 1983.
- [4] F.H. Jackson, A Basic Sine and Cosine with Symbolic Solutions of certain Differential Equations, Proc. Edin. Math. Soc. 22, 28-39, 1904.
- [5] S.Nalci and O.K.Pashaev, Q-Damped Oscillator and Degenerate Roots of Constant Coefficients q-Difference ODE, arXiv: 1107.2518, 2011.
- [6] S.Nalci and O.K.Pashaev, q-Bernoulli Numbers and Zeros of q -Sine Function (in preparation).
- [7] O.K. Pashaev and S.Nalci, Golden quantum oscillator and Binet-Fibonacci calculus, J.Phys.A:Math.Theor.45, 2012
- [8] S.Nalci and O.K.Pashaev, q-Travelling waves and D'Alembert solution of q-wave equation (in preparation).
- [9] O.K.Pashaev and S.Nalci, q-Holomorphic function as generalized analytic function (in preparation).